



The Open
University

MST124

Essential mathematics 1

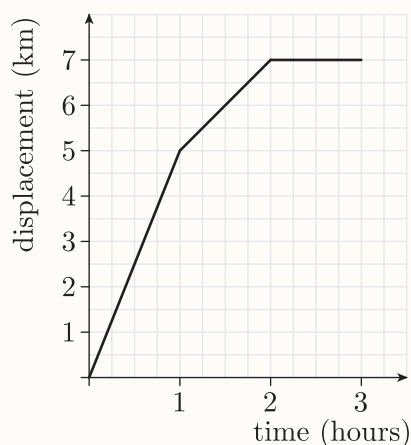
Exercise Booklet 6

1 What is differentiation?

The first two exercises review material from Unit 2.

Exercise 1

The displacement–time graph below represents a man’s walk along a straight path.



- How far does the man walk during the first hour?
- How far does he walk during the second hour?
- How far does he walk during the third hour?

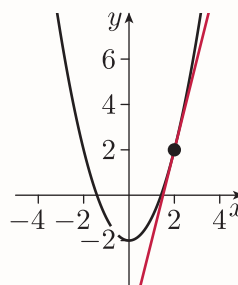
Exercise 2

For each of the following time periods, calculate the gradient of the section of the graph in Exercise 1 covering that time period.

- The first hour.
- The second hour.
- The third hour.

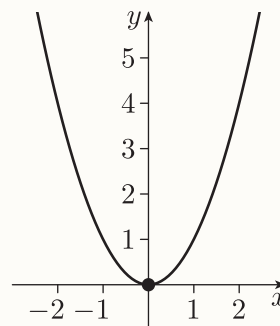
Exercise 3

Is the straight line in the figure below called a gradient or a tangent?



Exercise 4

In Subsection 1.3 of Unit 6 you saw that the gradient of the graph of $y = x^2$ is given by the formula $2x$. Use this formula to find the gradient of the graph at $x = 0$, then check your answer by looking at the graph below and working out rise/run for two points on the tangent to the graph at this point.



Exercise 5

Suppose that f is a function that is differentiable at x . Which of the following mean the same as ‘the value of the derivative of f at x ’? (You may choose more than one.)

- (a) The derived function of f .
- (b) The gradient of the graph of f at the point $(x, f(x))$.
- (c) f' .
- (d) The gradient of the tangent to the graph of f at the point $(x, f(x))$.
- (e) $f'(x)$.
- (f) The tangent at x .

Exercise 6

Suppose that f is a function. Which of the following mean the same as ‘the derivative of f ’? (You may choose more than one.)

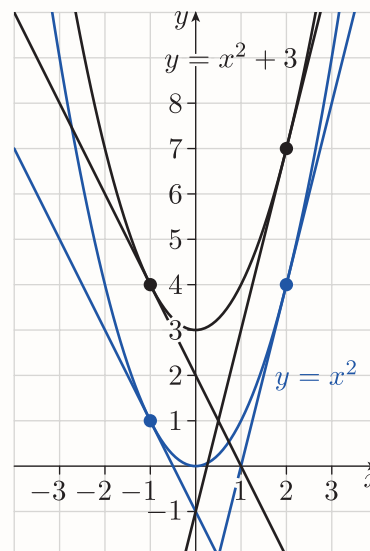
- (a) The value of the derivative of f at x .
- (b) The gradient of the graph of f at the point $(x, f(x))$.
- (c) f' .
- (d) The gradient of the tangent to the graph of f at the point $(x, f(x))$.
- (e) $f'(x)$.
- (f) The tangent to the graph of f at the point $(x, f(x))$.
- (g) The function that we obtain by differentiating f .

Exercise 7

Differentiate from first principles the function $f(x) = x^2 + 3$.

Exercise 8

The figure below shows the graphs of $f(x) = x^2$ and $f(x) = x^2 + 3$ drawn on the same axes.



- (a) What do you notice about the tangents to the graphs at the points with x -coordinates -1 and 2 ?
- (b) You saw in Subsection 1.3 of Unit 6 that the derivative of the function $f(x) = x^2$ is $f'(x) = 2x$. Compare this derivative with the derivative of the function $f(x) = x^2 + 3$, found in the solution to Exercise 7. What do you notice? Can you explain this by looking at the two graphs in the figure above?

Exercise 9

Differentiate from first principles the function $f(x) = (x + 3)^2$.

2 Finding derivatives of simple functions

Exercise 10

Differentiate the following functions.

(a) $f(x) = x^{17}$ (b) $f(x) = x^{22}$

(c) $g(x) = x^{-4}$ (d) $y = x^{-7}$

(e) $f(t) = \frac{1}{t^{14}}$ (f) $y = \frac{1}{x^{22}}$

(g) $h(x) = \frac{1}{x^{17}}$

Exercise 11

Differentiate the following functions.

(a) $f(x) = x^{7/3}$ (b) $y = x^{2/3}$

(c) $h(x) = x^{2/5}$ (d) $y = \frac{1}{x\sqrt{x}}$

(e) $f(t) = \sqrt[5]{t}$ (f) $f(x) = x^{7/2}$

(g) $g(t) = \frac{1}{t^{7/2}}$ (h) $y = \frac{1}{t\sqrt{t^3}}$

Exercise 12

- Find the gradients of the graph of the function $f(x) = 1/x^3$ at the points with x -coordinates 2 and -1 .
- Find the x -coordinate of the point where the graph of the function $f(x) = x^2$ has gradient $\frac{1}{2}$.
- At how many points does the graph of $f(x) = x^3$ have gradient 12?

Exercise 13

- Find $f'(x)$ for the function $f(x) = -\frac{1}{4}\sqrt{x}$.
- Find $\frac{dy}{dx}$ for the function $y = \sqrt{\frac{x}{4}}$.
- Differentiate the function $f(x) = \sqrt{\frac{16}{x}}$.
- Find the gradient of the graph of the function $f(t) = \frac{4}{t}$ at the point with t -coordinate -1 .

Exercise 14

Differentiate the following functions.

(a) $f(x) = \frac{3}{2}x^{10} - 2x^6 + 3x$

(b) $y = 7 + 4x^2 - 3x^4 + x^6$

(c) $f(t) = 3t^2 - 6t^3 + 9t^4 - 12$

(d) $u(t) = 12t - 2\pi t^2$

(e) $y = \ln 2 - x \ln 3 - x^2 \ln 4$

Exercise 15

Differentiate the following functions.

(a) $f(x) = \frac{2}{x} + 2x$

(b) $y = \frac{2}{x^2} - 2x^2 + \ln 2$

(c) $g(t) = (t-2)(t^2-3)$

(d) $y = \frac{x^2 - 2x - 4}{x^3}$

(e) $g(x) = \frac{7}{x} - x \ln 2 - e^2$

(f) $y(t) = \frac{\sqrt{t} - t}{t^2}$

Exercise 16

- What is the gradient of the graph of $y = \frac{1}{4}x^8 - \frac{1}{3}x^6 + \frac{1}{3}$ at the point $(2, 43)$?
- What is the gradient of the graph of $y = x^5 - 3x^3 - x^2 - 6$ at the point $(-1, -5)$?

Exercise 17

For each of the following quadratic functions f , write down an expression for $f'(x)$, and find the gradient of the graph of $y = f(x)$ at the point stated.

(a) $f(x) = x^2 - 7x$, $(9, 18)$

(b) $f(x) = 2 - 3x^2$, $(2, -10)$

(c) $f(x) = 7x^2 - 12x - 19$, $(1, -24)$

Exercise 18

The function f is given by
 $f(x) = 2x^2 - 5x + 1$.

- Write down an expression for $f'(x)$.
- Find the gradient of the graph of $y = f(x)$ at the point $(2, -1)$.
- Using a method from Unit 2, find the equation of the line that is the tangent to the graph at the point $(2, -1)$.
- Find the point on the graph of $y = f(x)$ at which the gradient is -9 .

3 Rates of change

Exercise 19

Suppose that a swimmer swims along a straight path, and her displacement s (in metres) at time t (in minutes) after she begins her swim is given by

$$s = 5t + \frac{1}{2}t^2.$$

Let the swimmer's velocity at time t be v (in metres per minute).

- Find an equation expressing v in terms of t .
- Hence find the swimmer's velocity after 10 minutes.

Exercise 20

Suppose that a ball is thrown vertically upwards, and its displacement s (in metres) from its starting point at time t (in seconds) after it was thrown is given by the equation $s = 15t - 5t^2$.

- Find the velocity v (in ms^{-1}) of the ball at time t .
- What is the velocity of the ball after 1 second?
- At what time does the ball have velocity zero?
- What is the height of the ball when its velocity is zero?

Exercise 21

A cat food company calculates that an appropriate formula for the total cost of producing its cat food is

$$c = 2000 + 3q + \frac{q^3}{5000},$$

where c is the weekly total cost (in pounds), and q is the weekly quantity produced (in kg).

- Use differentiation to find an equation for the marginal cost m in terms of the quantity q .
- What is the marginal cost, in pounds, of making extra cat food when the quantity already being produced is 150 kilograms?
- The company sells all the cat food that it makes at a price of £27 per kg. It decides to keep increasing production until the marginal cost of production is equal to the selling price. By writing down and solving a suitable equation, find the weekly quantity of cat food that the company should make.

4 Finding where functions are increasing, decreasing or stationary

Exercise 22

Consider the function
 $y = 2x^3 + 3x^2 - 12x - 7$.

- Find the derivative $\frac{dy}{dx}$ and factorise it.
- Show that y is increasing on each of the intervals $(-\infty, -2)$ and $(1, \infty)$, and decreasing on the interval $(-2, 1)$.

Exercise 23

- (a) Find the derivative of the function

$$g(x) = 3x^2 + 18x + 15.$$

- (b) Hence show that

$$g'(-3) = 0;$$

$$g'(x) < 0 \text{ for } x \text{ in } (-\infty, -3);$$

$$g'(x) > 0 \text{ for } x \text{ in } (-3, \infty).$$

- (c) Deduce from part (b) the intervals on which g is increasing or decreasing.
- (d) Find $g(0)$, $g(-3)$ and the values of x for which $g(x) = 0$.
- (e) Use the results from parts (c) and (d) to draw a rough sketch of the graph of this function.

Exercise 24

Find the stationary points of the following functions.

(a) $f(x) = 2x^3 - 9x^2 + 12x + 4$

(b) $g(t) = (t - 4)(t + 1)$

(c) $y = x^4 - 8x^3 + 2x^2$

Exercise 25

This question is about the function

$$f(x) = 4x^3 + 6x^2 - 9x - 1.$$

- (a) Find the stationary points of f .
- (b) Use the first derivative test to determine the nature of each of the stationary points.
- (c) Find the y -coordinate of each of the stationary points, and evaluate $f(0)$. Hence sketch the graph of f .

Exercise 26

Find the greatest and least values of the function

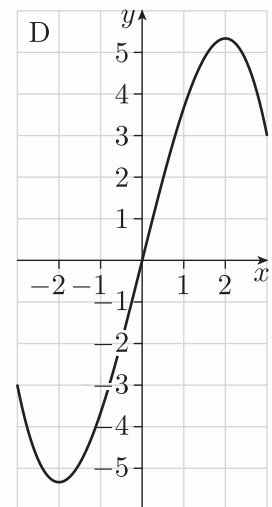
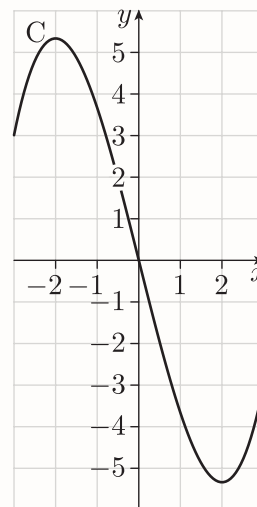
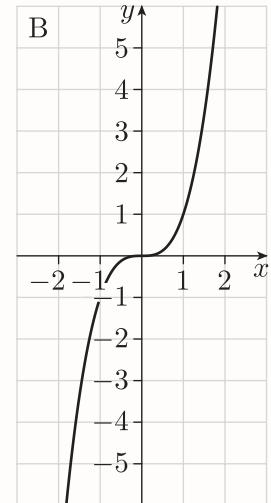
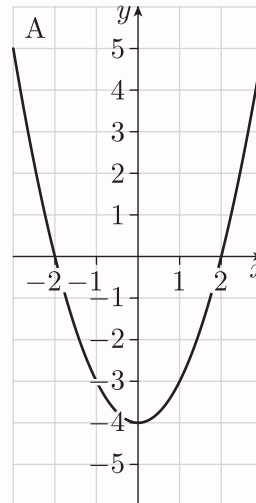
$$f(x) = x^3 - \frac{15}{2}x^2 + 12x + 2$$

on each of the following intervals.

- (a) $[0, 6]$ (b) $[-1, 3]$

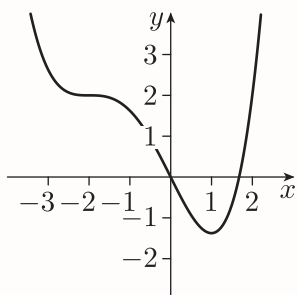
Exercise 27

Which of graphs A–D below could be the graph of a function f whose derivative is $f'(x) = x^2 - 4$?

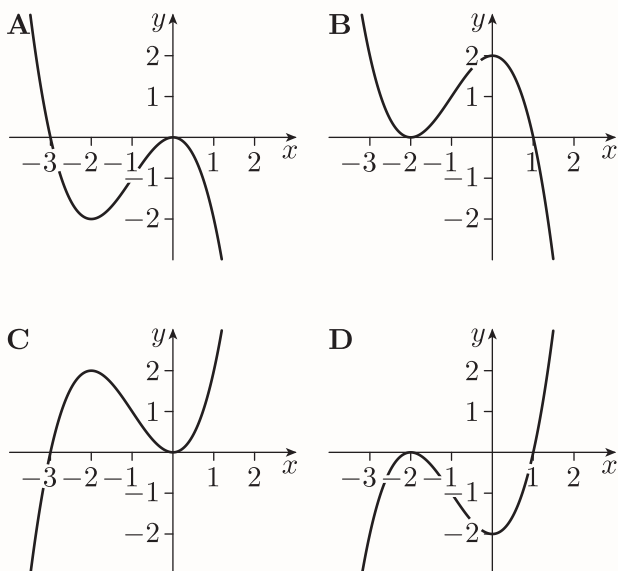


Exercise 28

The graph of a function f is shown below.



Which of the following graphs is the graph of the derivative of f ?



5 Differentiating twice

Exercise 29

This question is about the function

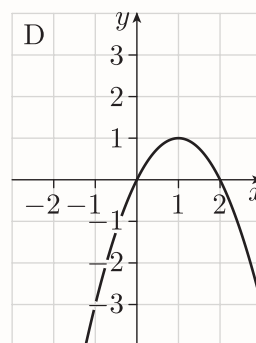
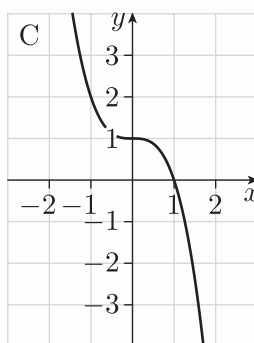
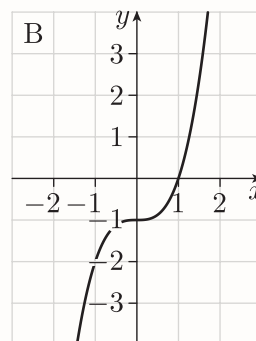
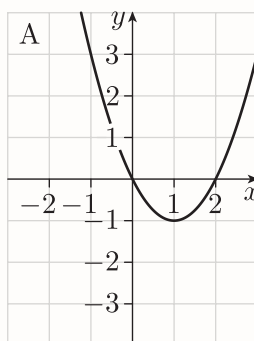
$$f(x) = x^4 - 8x^2 + 3.$$

- Find the stationary points of f .
- Use the second derivative test to determine the nature of each of the stationary points.
- Find the y -coordinate of each of the stationary points. Hence sketch the graph of f .

Exercise 30

For each of the following derivatives, determine which of graphs A–D below could be the graph of a function whose derivative it is. Do this by finding the second derivative and considering the values of x for which the second derivative is negative or positive.

- $\frac{dy}{dx} = 3x^2$
- $\frac{dy}{dx} = -3x^2$
- $\frac{dy}{dx} = 2x - 2$
- $\frac{dy}{dx} = -2x + 2$



Solutions to exercises

Solution to Exercise 1

- (a) In the first hour he walks 5 km.
 (b) In the second hour he walks 2 km.
 (c) In the third hour he walks 0 km.

Solution to Exercise 2

- (a) Choosing the points (1, 5) and (0, 0) gives

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{5 - 0}{1 - 0} = 5 \text{ km/h.}$$

- (b) Choosing the points (2, 7) and (1, 5) gives

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{7 - 5}{2 - 1} = 2 \text{ km/h.}$$

- (c) Choosing the points (3, 7) and (2, 7) gives

$$\text{gradient} = \frac{\text{rise}}{\text{run}} = \frac{7 - 7}{3 - 2} = 0 \text{ km/h.}$$

Solution to Exercise 3

The straight line is a tangent. A gradient is a number that would tell us how steep the tangent is.

Solution to Exercise 4

By the formula stated in the question, the gradient of the graph at $x = 0$ is $2 \times 0 = 0$.

The tangent at the point (0, 0) is the x -axis. Choosing the points (2, 0) and (-2, 0) on this line gives

$$\frac{\text{rise}}{\text{run}} = \frac{0 - 0}{2 - (-2)} = 0,$$

which confirms that the gradient is 0.

Solution to Exercise 5

Choices (b), (d) and (e) mean the same as 'the value of the derivative of f at x '.

Choices (a) and (c) are both functions, not values. Choice (f) is a line.

Solution to Exercise 6

Choices (c) and (g) mean the same as 'the derivative of f '.

Choices (a), (b) and (d) describe a value, not a function. Choice (e) is also a specific value, since $f'(x)$ is the value of f' at the point $(x, f(x))$. Choice (f) is the straight line that passes through the point $(x, f(x))$ and has the same gradient as the graph of f at this point.

Solution to Exercise 7

The difference quotient for the function $f(x) = x^2 + 3$ at x is

$$\begin{aligned} & \frac{f(x+h) - f(x)}{h} \\ &= \frac{((x+h)^2 + 3) - (x^2 + 3)}{h}. \end{aligned}$$

Multiplying out in the numerator gives

$$\frac{(x^2 + 2hx + h^2 + 3) - (x^2 + 3)}{h}$$

and simplifying gives

$$\begin{aligned} & \frac{x^2 + 2hx + h^2 + 3 - x^2 - 3}{h} \\ &= \frac{2hx + h^2}{h} \\ &= 2x + h. \end{aligned}$$

As h gets closer and closer to zero, the value of this expression gets closer and closer to $2x$.

So the formula for the derivative is

$$f'(x) = 2x.$$

Solution to Exercise 8

- (a) The two tangents drawn at $x = -1$ seem to be parallel, and the two tangents drawn at $x = 2$ seem to be parallel.
 (b) The derivatives of $f(x) = x^2$ and $f(x) = x^2 + 3$ are the same (they are both given by $f'(x) = 2x$). This makes sense, as the graph of $f(x) = x^2 + 3$ is obtained by translating the graph of $f(x) = x^2$ up by 3 units. This will not change the gradient of the tangent to the graph at any point.

Solution to Exercise 9

The difference quotient for the function $f(x) = (x+3)^2$ at x is

$$\frac{f(x+h) - f(x)}{h} = \frac{(x+h+3)^2 - (x+3)^2}{h}.$$

Multiplying out the first term in the numerator gives

$$\begin{aligned} & (x+h+3)^2 \\ &= (x+h+3)(x+h+3) \\ &= x(x+h+3) + h(x+h+3) + 3(x+h+3) \\ &= x^2 + hx + 3x + hx + h^2 + 3h + 3x + 3h + 9. \end{aligned}$$

Multiplying out the second term in the numerator gives

$$(x+3)^2 = x^2 + 6x + 9.$$

Hence the numerator is

$$(x+h+3)^2 - (x+3)^2 = 2hx + 6h + h^2.$$

So the difference quotient is

$$\frac{2hx + 6h + h^2}{h},$$

which simplifies to

$$2x + 6 + h.$$

As h gets closer and closer to zero, the value of this expression gets closer and closer to $2x + 6$.

So the formula for the derivative is $f'(x) = 2x + 6$.

Solution to Exercise 10

In each part we use the formula for the derivative of a power function, which is

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

(a) Here $f(x) = x^{17}$, so

$$f'(x) = 17x^{17-1} = 17x^{16}.$$

(b) Here $f(x) = x^{22}$, so

$$f'(x) = 22x^{22-1} = 22x^{21}.$$

(c) Here $g(x) = x^{-4}$, so

$$g'(x) = -4x^{-4-1} = -4x^{-5},$$

which can also be written as $g'(x) = -\frac{4}{x^5}$.

(d) Here $y = x^{-7}$, so

$$\frac{dy}{dx} = -7x^{-7-1} = -7x^{-8},$$

which can also be written as $\frac{dy}{dx} = -\frac{7}{x^8}$.

(e) Here $f(t) = \frac{1}{t^{14}} = t^{-14}$, so

$$f'(t) = -14t^{-14-1} = -14t^{-15} = -\frac{14}{t^{15}}.$$

(f) Here $y = \frac{1}{x^{22}} = x^{-22}$, so

$$\frac{dy}{dx} = -22x^{-22-1} = -22x^{-23} = -\frac{22}{x^{23}}.$$

(g) Here $h(x) = \frac{1}{x^{17}} = x^{-17}$, so

$$h'(x) = -17x^{-17-1} = -17x^{-18} = -\frac{17}{x^{18}}.$$

Solution to Exercise 11

In each part we use the formula for the derivative of a power function as in Exercise 10.

(a) Here $f(x) = x^{7/3}$, so

$$f'(x) = \frac{7}{3}x^{(7/3)-1} = \frac{7}{3}x^{4/3}.$$

(b) Here $y = x^{2/3}$, so

$$\frac{dy}{dx} = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

(c) Here $h(x) = x^{2/5}$, so

$$h'(x) = \frac{2}{5}x^{(2/5)-1} = \frac{2}{5}x^{-3/5} = \frac{2}{5x^{3/5}}.$$

(d) Here $y = \frac{1}{x\sqrt{x}} = \frac{1}{x^{3/2}} = x^{-3/2}$, so

$$\begin{aligned}\frac{dy}{dx} &= -\frac{3}{2}x^{(-3/2)-1} \\ &= -\frac{3}{2}x^{-5/2} \\ &= -\frac{3}{2x^2\sqrt{x}}.\end{aligned}$$

(The final answer can also be left as $-\frac{3}{2x^{5/2}}$.)

(e) Here $f(t) = \sqrt[5]{t} = t^{1/5}$ so

$$f'(t) = \frac{1}{5}t^{(1/5)-1} = \frac{1}{5}t^{-4/5} = \frac{1}{5t^{4/5}}.$$

(f) Here $f(x) = x^{7/2}$, so

$$f'(x) = \frac{7}{2}x^{(7/2)-1} = \frac{7}{2}x^{5/2}.$$

(g) Here $g(t) = \frac{1}{t^{7/2}} = t^{-7/2}$, so

$$\begin{aligned}g'(t) &= -\frac{7}{2}t^{(-7/2)-1} \\ &= -\frac{7}{2}t^{-9/2} \\ &= -\frac{7}{2t^{9/2}}.\end{aligned}$$

(h) Here

$$y = \frac{1}{t\sqrt{t^3}} = \frac{1}{t^{5/2}} = t^{-5/2},$$

so

$$\begin{aligned}\frac{dy}{dt} &= -\frac{5}{2}t^{(-5/2)-1} \\ &= -\frac{5}{2}t^{-7/2} \\ &= -\frac{5}{2t^3\sqrt{t}}.\end{aligned}$$

(The final answer can also be left as $-\frac{5}{2t^{7/2}}$.)

Solution to Exercise 12

(a) Here $f(x) = \frac{1}{x^3} = x^{-3}$, so

$$f'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}.$$

So the gradient of the graph of f at the point with x -coordinate 2 is

$$f'(2) = -\frac{3}{2^4} = -\frac{3}{16},$$

and the gradient at the point with x -coordinate -1 is

$$f'(-1) = -\frac{3}{(-1)^4} = -3.$$

(b) Here $f(x) = x^2$, so $f'(x) = 2x$. So the graph of f has gradient $\frac{1}{2}$ when x satisfies the equation $2x = \frac{1}{2}$; that is, when $x = \frac{1}{4}$. So the required x -coordinate is $\frac{1}{4}$.

(c) Here $f(x) = x^3$, so $f'(x) = 3x^2$. So the graph of f has gradient 12 when x satisfies the equation $3x^2 = 12$; that is, when $x^2 = 4$.

There are two values of x for which $x^2 = 4$ (namely 2 and -2), so there are two points at which the graph of $f(x) = x^3$ has gradient 12.

Solution to Exercise 13

(a) Here

$$f(x) = -\frac{1}{4}\sqrt{x} = -\frac{1}{4}x^{1/2},$$

so

$$\begin{aligned} f'(x) &= -\frac{1}{4} \times \frac{1}{2}x^{-1/2} \\ &= -\frac{1}{8}x^{-1/2} \\ &= -\frac{1}{8\sqrt{x}}. \end{aligned}$$

(b) Here

$$y = \sqrt{\frac{x}{4}} = \frac{\sqrt{x}}{\sqrt{4}} = \frac{\sqrt{x}}{2} = \frac{1}{2}x^{1/2}.$$

(Remember that $\sqrt{x}$ denotes the positive square root.) So

$$\frac{dy}{dx} = \frac{1}{2} \times \frac{1}{2}x^{-1/2} = \frac{1}{4}x^{-1/2} = \frac{1}{4\sqrt{x}}.$$

(c) Here

$$\sqrt{\frac{16}{x}} = \frac{4}{\sqrt{x}} = 4x^{-1/2},$$

so

$$f'(x) = 4 \times \left(-\frac{1}{2}\right)x^{-3/2} = -\frac{2}{x^{3/2}}.$$

(d) Here $f(t) = \frac{4}{t} = 4t^{-1}$, so

$$f'(t) = 4 \times (-1) \times t^{-2} = -\frac{4}{t^2}.$$

When $t = -1$, we have

$$f'(-1) = -\frac{4}{(-1)^2} = -\frac{4}{1} = -4.$$

That is, the gradient of the graph of the function $f(t) = 4/t$ at the point with t -coordinate -1 is -4 .

Solution to Exercise 14

In each part we use the constant multiple rule and the sum rule, together with the formula for the derivative of a power function.

(a) Here $f(x) = \frac{3}{2}x^{10} - 2x^6 + 3x$, so

$$\begin{aligned} f'(x) &= \frac{3}{2} \times 10x^9 - 2 \times 6x^5 + 3 \times 1 \\ &= 15x^9 - 12x^5 + 3. \end{aligned}$$

(b) Here $y = 7 + 4x^2 - 3x^4 + x^6$, so

$$\begin{aligned} \frac{dy}{dx} &= 4 \times 2x - 3 \times 4x^3 + 6x^5 \\ &= 8x - 12x^3 + 6x^5. \end{aligned}$$

(c) Here $f(t) = 3t^2 - 6t^3 + 9t^4 - 12$, so

$$\begin{aligned} f'(t) &= 2 \times 3t - 3 \times 6t^2 + 4 \times 9t^3 \\ &= 6t - 18t^2 + 36t^3. \end{aligned}$$

(d) Here $u(t) = 12t - 2\pi t^2$, so

$$\begin{aligned} u'(t) &= 12 - 2 \times 2\pi t \\ &= 12 - 4\pi t. \end{aligned}$$

(e) Here $y = \ln 2 - x \ln 3 - x^2 \ln 4$, and $\ln 2$, $\ln 3$ and $\ln 4$ are simply constants, so

$$\frac{dy}{dx} = -\ln 3 - 2x \ln 4.$$

Solution to Exercise 15

In each part we use the constant multiple rule and the sum rule, together with the formula for the derivative of a power function.

(a) Here

$$\begin{aligned} f(x) &= \frac{2}{x} + 2x \\ &= 2x^{-1} + 2x, \end{aligned}$$

so

$$\begin{aligned} f'(x) &= -1 \times 2x^{-2} + 2 \\ &= -2x^{-2} + 2 \\ &= -\frac{2}{x^2} + 2 \\ &= 2 - \frac{2}{x^2}. \end{aligned}$$

(b) Here

$$\begin{aligned} y &= \frac{2}{x^2} - 2x^2 + \ln 2 \\ &= 2x^{-2} - 2x^2 + \ln 2, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= -2 \times 2x^{-3} - 2 \times 2x \\ &= -\frac{4}{x^3} - 4x. \end{aligned}$$

(c) Here

$$\begin{aligned} g(t) &= (t-2)(t^2-3) \\ &= t^3 - 2t^2 - 3t + 6, \end{aligned}$$

so

$$g'(t) = 3t^2 - 4t - 3.$$

(d) Here

$$\begin{aligned} y &= \frac{x^2 - 2x - 4}{x^3} \\ &= \frac{x^2}{x^3} - \frac{2x}{x^3} - \frac{4}{x^3} \\ &= \frac{1}{x} - \frac{2}{x^2} - \frac{4}{x^3} \\ &= x^{-1} - 2x^{-2} - 4x^{-3}, \end{aligned}$$

so

$$\begin{aligned} \frac{dy}{dx} &= (-1)x^{-2} - (-2) \times 2x^{-3} \\ &\quad - (-3) \times 4x^{-4} \\ &= -\frac{1}{x^2} + \frac{4}{x^3} + \frac{12}{x^4}. \end{aligned}$$

(e) Here

$$\begin{aligned} g(x) &= \frac{7}{x} - x \ln 2 - e^2 \\ &= 7x^{-1} - x \ln 2 - e^2, \end{aligned}$$

so

$$\begin{aligned} g'(x) &= -7x^{-2} - \ln 2 \\ &= -\frac{7}{x^2} - \ln 2. \end{aligned}$$

(f) Here

$$\begin{aligned} y(t) &= \frac{\sqrt{t} - t}{t^2} \\ &= \frac{t^{1/2}}{t^2} - \frac{t}{t^2} \\ &= t^{-3/2} - t^{-1}, \end{aligned}$$

so

$$\begin{aligned} y'(t) &= -\frac{3}{2}t^{-5/2} - (-1)t^{-2} \\ &= -\frac{3}{2t^{5/2}} + \frac{1}{t^2} \\ &= \frac{1}{t^2} - \frac{3}{2t^{5/2}}. \end{aligned}$$

Solution to Exercise 16

(a) The derivative of the function

$$y = \frac{1}{4}x^8 - \frac{1}{3}x^6 + \frac{1}{3}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{4} \times 8x^7 - \frac{1}{3} \times 6x^5 \\ &= 2x^7 - 2x^5. \end{aligned}$$

The point (2, 43) on the graph corresponds to $x = 2$, for which the gradient is

$$\begin{aligned} \frac{dy}{dx} &= 2 \times 2^7 - 2 \times 2^5 \\ &= 256 - 64 = 192. \end{aligned}$$

(b) The derivative of the function

$$f(x) = x^5 - 3x^3 - x^2 - 6$$

$$\begin{aligned} \frac{dy}{dx} &= 5x^4 - 3 \times 3x^2 - 2x \\ &= 5x^4 - 9x^2 - 2x. \end{aligned}$$

The point (-1, -5) on the graph corresponds to $x = -1$, for which the gradient is

$$\begin{aligned} \frac{dy}{dx} &= 5(-1)^4 - 9(-1)^2 - 2(-1) \\ &= 5 - 9 + 2 = -2. \end{aligned}$$

Solution to Exercise 17

- (a) Here $f(x) = x^2 - 7x$, so $f'(x) = 2x - 7$.
Hence the gradient at the point $(9, 18)$, where $x = 9$, is $f'(9) = 18 - 7 = 11$.
- (b) Here $f(x) = 2 - 3x^2$, so $f'(x) = -6x$.
Hence the gradient at the point $(2, -10)$, where $x = 2$, is $f'(2) = -12$.
- (c) Here $f(x) = 7x^2 - 12x - 19$, so $f'(x) = 14x - 12$.
Hence the gradient at the point $(1, -24)$, where $x = 1$, is $f'(1) = 14 - 12 = 2$.

Solution to Exercise 18

- (a) We have $f(x) = 2x^2 - 5x + 1$, so $f'(x) = 4x - 5$.
- (b) The gradient of the graph at the point $(2, -1)$, where $x = 2$, is $f'(2) = 3$.
- (c) The tangent passes through the point $(2, -1)$, and its gradient is the gradient of the graph at the point $(2, -1)$, which is $f'(2) = 3$.

Hence the equation of the tangent is

$$y - (-1) = 3(x - 2),$$

which can be rearranged as $y = 3x - 7$.

(This solution uses the fact that the equation of a line with gradient m that passes through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$, from Unit 2.)

- (d) The gradient at the point (x, y) is $f'(x) = 4x - 5$. So the gradient is -9 when $4x - 5 = -9$; that is, when $x = -1$. The corresponding value of y is

$$\begin{aligned} y &= f(-1) = 2(-1)^2 - 5(-1) + 1 \\ &= 2 + 5 + 1 = 8. \end{aligned}$$

So the required point is $(-1, 8)$.

Solution to Exercise 19

- (a) The swimmer's displacement s at time t is given by

$$s = 5t + \frac{1}{2}t^2.$$

Hence her velocity v at time t is given by

$$v = \frac{ds}{dt} = 5 + t.$$

That is, the required equation is

$$v = 5 + t.$$

- (b) When $t = 10$, we have $v = 5 + 10$, so the swimmer's velocity after 10 minutes is 15 metres per minute.

Solution to Exercise 20

- (a) The ball's displacement s at time t is given by

$$s = 15t - 5t^2.$$

Hence its velocity v at time t is given by:

$$v = \frac{ds}{dt} = 15 - 10t.$$

That is, the required equation is

$$v = 15 - 10t.$$

- (b) When $t = 1$, we have $v = 15 - 10 = 5$, so the velocity after 1 second is 5 m s^{-1} .
- (c) When $v = 0$, we have $0 = 15 - 10t$, so $t = \frac{3}{2}$. So the ball has velocity zero after 1.5 seconds.
- (d) From part (c), $v = 0$ when $t = \frac{3}{2}$. At this time,

$$\begin{aligned} s &= 15 \times \frac{3}{2} - 5 \times \left(\frac{3}{2}\right)^2 \\ &= \frac{45}{2} - \frac{45}{4} \\ &= \frac{45}{4} \\ &= 11.25. \end{aligned}$$

So the height of the ball when its velocity is zero is 11.25 metres.

Solution to Exercise 21

- (a) The equation for c in terms of q is

$$c = 2000 + 3q + \frac{q^3}{5000}.$$

Hence

$$m = \frac{dc}{dq} = 3 + \frac{3q^2}{5000}.$$

- (b) When the quantity of cat food already being produced is 150 kilograms, the marginal cost is given by

$$m = 3 + \frac{3 \times 150^2}{5000} = 16.5,$$

so it is £16.50.

- (c) The marginal cost is equal to the selling price when $m = 27$. From part (a), the value of q for which this happens is given by

$$3 + \frac{3q^2}{5000} = 27.$$

Solving this equation gives

$$3 + \frac{3q^2}{5000} = 27$$

$$\frac{3q^2}{5000} = 24$$

$$q^2 = \frac{24 \times 5000}{3}$$

$$q^2 = 40\,000$$

$$q = \pm 200$$

$$q = 200$$

(since q is positive).

So the company should make 200 kilograms of cat food per week.

Solution to Exercise 22

(a) The derivative is

$$\begin{aligned}\frac{dy}{dx} &= 6x^2 + 6x - 12 \\ &= 6(x^2 + x - 2) \\ &= 6(x+2)(x-1).\end{aligned}$$

(b) When x is less than -2 , the values of $(x+2)$ and $(x-1)$ are both negative. Hence the value of $dy/dx = 6(x+2)(x-1)$ is positive.

When x is greater than 1 , the values of $(x+2)$ and $(x-1)$ are both positive. Hence the value of $dy/dx = 6(x+2)(x-1)$ is positive.

When x is in the interval $(-2, 1)$, the value of $(x+2)$ is positive and the value of $(x-1)$ is negative. Hence the value of $dy/dx = 6(x+2)(x-1)$ is negative.

Therefore, by the increasing/decreasing criterion, the given function is increasing on each of the intervals $(-\infty, -2)$ and $(1, \infty)$, and decreasing on the interval $(-2, 1)$.

Solution to Exercise 23

(a) The derivative of the function

$$g(x) = 3x^2 + 18x + 15$$

$$\text{is } g'(x) = 6x + 18 = 6(x+3).$$

(b) We have $g'(-3) = 6(-3+3) = 0$.

For $x < -3$, we have $x+3 < 0$, so $g'(x) = 6(x+3)$ is the product of 6 and a negative number, and hence $g'(x) < 0$.

For $x > -3$, we have $x+3 > 0$, so $g'(x) = 6(x+3)$ is the product of 6 and a positive number, and hence $g'(x) > 0$.

(c) By the increasing/decreasing criterion, g is increasing on $(-3, \infty)$ and decreasing on $(-\infty, -3)$.

(d) We have $g(0) = 15$, and

$$g(-3) = 3(-3)^2 + 18(-3) + 15 = -12.$$

Also, solving the equation $g(x) = 0$ gives

$$3x^2 + 18x + 15 = 0$$

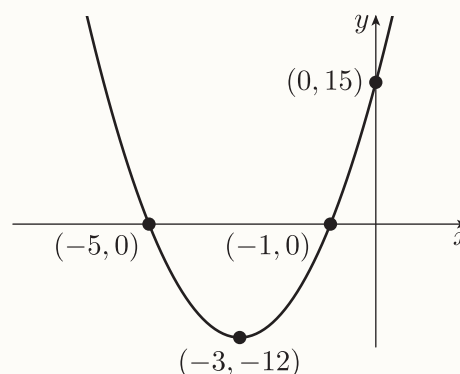
$$x^2 + 6x + 5 = 0$$

$$(x+1)(x+5) = 0$$

$$x = -1 \text{ or } x = -5.$$

So $g(x) = 0$ when $x = -1$ or $x = -5$.

(e) By the results from parts (c) and (d), the graph passes through the points $(-5, 0)$, $(-3, -12)$, $(-1, 0)$ and $(0, 15)$. It is decreasing for $x < -3$ and increasing for $x > -3$, so it takes its least value, -12 , at $x = -3$. A sketch of the graph is below.



Solution to Exercise 24

(a) The function is

$$f(x) = 2x^3 - 9x^2 + 12x + 4.$$

Its derivative is

$$\begin{aligned}f'(x) &= 6x^2 - 18x + 12 \\ &= 6(x^2 - 3x + 2) \\ &= 6(x-1)(x-2).\end{aligned}$$

Solving the equation $f'(x) = 0$ gives

$$6(x-1)(x-2) = 0;$$

that is,

$$x = 1 \text{ or } x = 2.$$

Hence the stationary points are 2 and 1.

(b) The function is

$$\begin{aligned}g(t) &= (t-4)(t+1) \\ &= t^2 - 3t - 4.\end{aligned}$$

Its derivative is

$$g'(t) = 2t - 3.$$

Solving the equation $g'(t) = 0$ gives

$$2t - 3 = 0;$$

that is,

$$t = \frac{3}{2}.$$

Hence the stationary point is $\frac{3}{2}$.

(c) The function is

$$y = x^4 - 8x^3 + 2x^2.$$

Its derivative is

$$\begin{aligned}\frac{dy}{dx} &= 4x^3 - 24x^2 + 4x \\ &= 4x(x^2 - 6x + 1).\end{aligned}$$

(The quadratic in the brackets has discriminant

$$(-6)^2 - 4 \times 1 \times 1 = 36 - 4 = 32,$$

which is not a perfect square, so the quadratic cannot be factorised using integers.)

Solving the equation $dy/dx = 0$ gives

$$4x(x^2 - 6x + 1) = 0;$$

that is,

$$x = 0 \quad \text{or} \quad x^2 - 6x + 1 = 0.$$

Using the quadratic formula to solve the quadratic equation above gives

$$\begin{aligned}x &= \frac{6 \pm \sqrt{32}}{2} \\ &= \frac{6 \pm 4\sqrt{2}}{2} \\ &= 3 \pm 2\sqrt{2}.\end{aligned}$$

Hence the stationary points are 0, $3 - 2\sqrt{2}$ and $3 + 2\sqrt{2}$.

Solution to Exercise 25

(a) The derivative of the function

$$f(x) = 4x^3 + 6x^2 - 9x - 1$$

$$\begin{aligned}f'(x) &= 12x^2 + 12x - 9 \\ &= 3(4x^2 + 4x - 3) \\ &= 3(2x - 1)(2x + 3).\end{aligned}$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are $x = \frac{1}{2}$ and $x = -\frac{3}{2}$.

(b) We can determine the natures of the stationary points by choosing sample points.

Consider the values -2 , 0 and 1 . The values -2 and 0 lie on each side of the stationary point $-\frac{3}{2}$, and the values 0 and 1 lie on each side of the stationary point $\frac{1}{2}$ (with no further stationary points in between, in each case).

Also, f is differentiable at every value of x (since it is a polynomial function).

We have

$$f'(-2) = 48 - 24 - 9 = 15 > 0,$$

$$f'(0) = -9 < 0,$$

$$f'(1) = 12 + 12 - 9 = 15 > 0.$$

So, by the first derivative test, f has a local maximum at $x = -\frac{3}{2}$, and it has a local minimum at $x = \frac{1}{2}$.

(An alternative method for part (b) is to use a table of signs.)

(c) The y -coordinates of the stationary points are

$$\begin{aligned}f\left(-\frac{3}{2}\right) &= 4\left(-\frac{27}{8}\right) + 6\left(\frac{9}{4}\right) - 9\left(-\frac{3}{2}\right) - 1 \\ &= \frac{25}{2},\end{aligned}$$

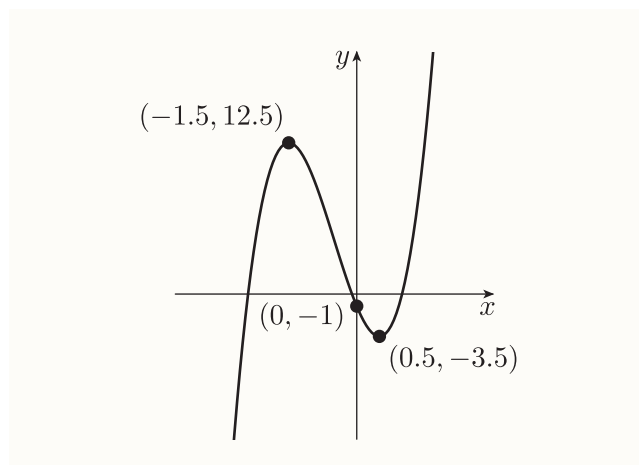
and

$$\begin{aligned}f\left(\frac{1}{2}\right) &= 4\left(\frac{1}{8}\right) + 6\left(\frac{1}{4}\right) - 9\left(\frac{1}{2}\right) - 1 \\ &= -\frac{7}{2}.\end{aligned}$$

Also $f(0) = -1$.

Hence the graph passes through the point $\left(-\frac{3}{2}, \frac{25}{2}\right)$, a local maximum, and the point $\left(\frac{1}{2}, -\frac{7}{2}\right)$, a local minimum. Also, its y -intercept is -1 .

A sketch of the graph is below.



Solution to Exercise 26

- (a) We follow the steps of the strategy given in Subsection 4.3 of Unit 6.

The derivative of the function

$$f(x) = x^3 - \frac{15}{2}x^2 + 12x + 2 \text{ is}$$

$$\begin{aligned} f'(x) &= 3x^2 - 15x + 12 \\ &= 3(x^2 - 5x + 4) \\ &= 3(x-1)(x-4). \end{aligned}$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are at $x = 1$ and $x = 4$.

Both stationary points are inside the interval $[0, 6]$, and the values of f at these points are

$$\begin{aligned} f(1) &= 1 - \frac{15}{2} + 12 + 2 = \frac{15}{2}, \\ f(4) &= 64 - 120 + 48 + 2 = -6. \end{aligned}$$

At the endpoints of the interval $[0, 6]$, the values of f are

$$\begin{aligned} f(0) &= 2, \\ f(6) &= 216 - 270 + 72 + 2 = 20. \end{aligned}$$

The greatest value of $f(x)$ on $[0, 6]$ is 20 (at $x = 6$), and the least value is -6 (at $x = 4$).

- (b) From part (a) we know that the stationary points of f are $x = 1$ and $x = 4$. Only one of these stationary points, $x = 1$, is inside the interval $[-1, 3]$. From part (a),

$$f(1) = \frac{15}{2}.$$

At the endpoints of the interval $[-1, 3]$, the values of f are

$$\begin{aligned} f(-1) &= -1 - \frac{15}{2} - 12 + 2 = -\frac{37}{2}, \\ f(3) &= 27 - \frac{135}{2} + 36 + 2 = -\frac{5}{2}. \end{aligned}$$

The greatest value of $f(x)$ on $[-1, 3]$ is $\frac{15}{2}$ (at $x = 1$), and the least value is $-\frac{37}{2}$ (at $x = -1$).

Solution to Exercise 27

We have

$$f'(x) = x^2 - 4 = (x+2)(x-2).$$

So f has stationary points at $x = -2$ and $x = 2$, and hence its graph must be either graph C or graph D.

Also $f'(0) = -4$, so the graph has a negative gradient when $x = 0$. Hence the correct graph is graph C.

(In fact, the equation $f'(0) = -4$ alone is enough to allow you to identify graph C as the correct graph. Only graph C appears to have a gradient of about -4 at $x = 0$.)

Solution to Exercise 28

The gradient of the graph of the function f appears to be negative or zero when $x < 1$, and positive when $x > 1$. So the values taken by the graph of the derivative of f must be negative or zero when $x < 1$, and positive when $x > 1$. Hence the graph of the derivative of f is graph D.

Solution to Exercise 29

- (a) The derivative of the function

$$f(x) = x^4 - 8x^2 + 3 \text{ is}$$

$$\begin{aligned} f'(x) &= 4x^3 - 16x \\ &= 4x(x^2 - 4) \\ &= 4x(x+2)(x-2). \end{aligned}$$

Solving the equation $f'(x) = 0$, we find that the stationary points of f are $x = -2$, $x = 0$ and $x = 2$.

- (b) The second derivative of f is

$$f''(x) = 12x^2 - 16.$$

For the stationary point $x = -2$, we have

$$f''(-2) = 12(-2)^2 - 16 = 32 > 0.$$

Hence, by the second derivative test, f has a local minimum at $x = -2$.

For the stationary point $x = 0$, we have

$$f''(0) = -16 < 0.$$

Hence, by the second derivative test, f has a local maximum at $x = 0$.

For the stationary point $x = 2$, we have

$$f''(2) = 12(2)^2 - 16 = 32 > 0.$$

Hence, by the second derivative test, f has a local minimum at $x = 2$.

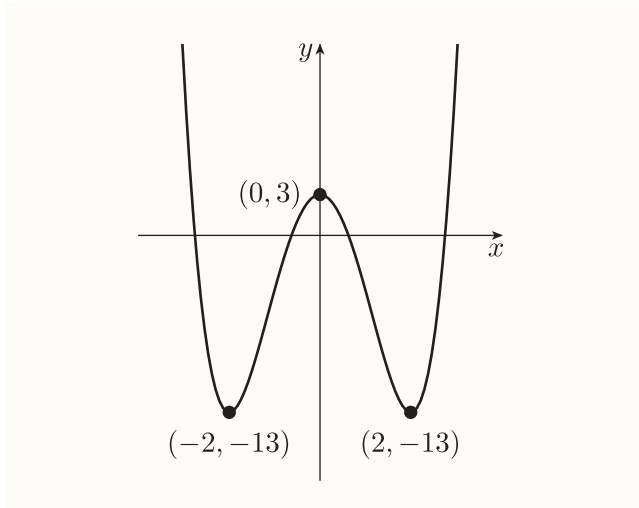
- (c) The y -coordinates of the stationary points on the graph of $y = f(x)$ are, respectively,

$$\begin{aligned} f(-2) &= (-2)^4 - 8(-2)^2 + 3 = -13, \\ f(0) &= 3, \\ f(2) &= 2^4 - 8(2)^2 + 3 = -13. \end{aligned}$$

Hence the graph passes through the points $(-2, -13)$, a local minimum, $(0, 3)$, a local maximum, and $(2, -13)$, a local minimum.

Also $f(0) = 3$, so its y -intercept is 3.

A sketch of the graph is below.



Solution to Exercise 30

- (a) Here $\frac{dy}{dx} = 3x^2$, so $\frac{d^2y}{dx^2} = 6x$.

Since d^2y/dx^2 is negative when $x < 0$ and positive when $x > 0$, the graph of the function is concave down when $x < 0$ and concave up when $x > 0$. So the correct graph is graph B.

- (b) Here $\frac{dy}{dx} = -3x^2$, so $\frac{d^2y}{dx^2} = -6x$.

Since d^2y/dx^2 is positive when $x < 0$ and negative when $x > 0$, the graph of the function is concave up when $x < 0$ and concave down when $x > 0$. So the correct graph is graph C.

- (c) Here $\frac{dy}{dx} = 2x - 2$, so $\frac{d^2y}{dx^2} = 2$.

Since d^2y/dx^2 is always positive, the graph of the function is always concave up. So the correct graph is graph A.

- (d) Here $\frac{dy}{dx} = -2x + 2$, so $\frac{d^2y}{dx^2} = -2$.

Since d^2y/dx^2 is always negative, the graph of the function is always concave down. So the correct graph is graph D.